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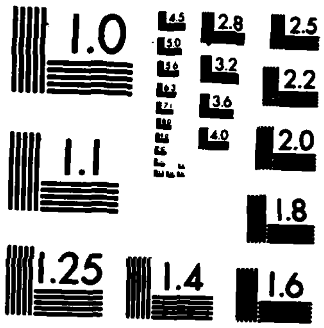
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MRC Technical Summary Report #2728

SELF SIMILAR SOLUTIONS FOR A DEGENERATE CAUCHY PROBLEM

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AD-A147 241

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August 1984

(Received June 14, 1984)

Approved for public release
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SELF SIMILAR SOLUTIONS FOR A DEGENERATE CAUCHY PROBLEM

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ABSTRACT

We determine the self-similar solutions of the Cauchy problem

$$\begin{aligned} v_t &= \phi(v)_{xx}, \quad x \in \mathbb{R},\ t > 0 , \\ v(x,0) &= g(x) \end{aligned} \tag{P}$$

for the constitutive function $\phi(v) = \max(0,v)$ and the model datum

$$g(x) = \begin{cases} (p_+)x^\gamma, & x > 0 \\ -(p_-)|x|^\gamma, & x < 0 \end{cases} \tag{D}$$

where $\gamma, p_\pm > 0$. It is shown that the unique solution of (P), (D) is

$$v(x,t) = \begin{cases} -(p_-)|x|^\gamma, & x < -\kappa\sqrt{t} \\ t^{\gamma/2}\psi(\frac{x}{\sqrt{t}}), & x > -\kappa\sqrt{t} \end{cases}$$

where

$$\psi(\xi) = [b_1(\kappa)D_{-\gamma-1}(\frac{\xi}{\sqrt{2}}) + b_2(\kappa)D_\gamma(\frac{i\xi}{\sqrt{2}})]\exp(-\frac{\xi^2}{8}) ,$$

$$b_1(\kappa) = \frac{p_-\kappa^{\gamma+1}D_\gamma(-\frac{i\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8})}{i\sqrt{2}\exp(\frac{\gamma+1}{2}\pi i)} ,$$

$$b_2(\kappa) = \frac{p_-\kappa^{\gamma+1}D_{-\gamma-1}(-\frac{\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8})}{i\sqrt{2}\exp(\pi i\frac{\gamma+1}{2})}$$

and κ is implicitly determined by the equation

$$p := p_+/p_- = (\frac{\kappa}{\sqrt{2}})^{\gamma+1} D_{-\gamma-1}(-\frac{\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8}) ;$$

$D_\nu(\cdot)$ denotes the parabolic cylinder function of index ν.

AMS (MOS) Subject Classifications: 35K55, 34B30, 35R35

Key Words: Cauchy problem, degenerate, self-similar, free boundary, parabolic cylinder functions

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

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SIGNIFICANCE AND EXPLANATION

Degenerate parabolic equations arise in the description of melting processes, gas dynamics and certain biological models. The interfaces corresponding to degeneracies in the constitutive function usually separate different media in the physical problem.

The particular problem stated in the abstract is related to nonlinear diffusion equations with nonmonotone constitutive functions, as has been discussed in [HN1-3]. In this report ~~we~~ the authors obtain self-similar solutions for (P) for a class of model initial data. The qualitative behavior of these solutions, in particular of their interfaces, is typical of the situation in more general problems. In a subsequent report with Vazquez [HNV] they we use such self-similar solutions as comparison functions to study the regularity and the behavior for small time of the interfaces for problem (P) with

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$\phi(v) = \max(0, \phi(v))$ where ϕ is strictly monotone increasing.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SELF SIMILAR SOLUTIONS FOR A DEGENERATE CAUCHY PROBLEM

Klaus Höllig and John A. Nohel

1. INTRODUCTION AND RESULT. In this note we study certain aspects of the degenerate Cauchy problem

$$\begin{aligned} &v_t = \phi(v)_{xx}, \quad x \in \mathbb{R},\ t > 0\ , \\ &v(x,0) = g(x) \end{aligned} \tag{P}$$

for the constitutive function $\phi(v) = \max(v,0)$. We assume that the initial data g are smooth on $\mathbb{R}\setminus\{0\}$ with at most polynomial growth at infinity and satisfy

$$xg(x) > 0, \quad x \neq 0\ ,$$
$$g(0) = 0\ .$$

Problems of this type arise as convexifications of diffusion equations with nonmonotone constitutive functions as has been discussed in [HN1]. The behavior of solutions for (P) is similar to the one phase Stefan problem where $g(x) \equiv -1$ for $x < 0$. Existence and uniqueness of weak solutions of (P) follows from nonlinear semigroup theory [BCP,E]. Moreover, using standard approximation arguments one can show the existence of a continuous monotone decreasing free boundary $t \to s(t)$ where $v(s(t)^+,t) = 0$.

The pair (v,s) satisfies the free boundary problem

$$\begin{aligned} &v_t = v_{xx}, \quad x > s(t),\ t > 0\ , \\ &v(x,0) = g(x), \quad x > 0\ , \\ &v(s(t)^+,t) = 0\ , \\ &g(s(t))s'(t) = v_x(s(t)^+,t)\ , \\ &s(0) = 0\ . \end{aligned} \tag{P̃}$$

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Conversely, the solution v of $(\bar{P})$ extended by $v(x,t) = g(x)$ for $x < s(t)$ is a weak solution of (P).

We are interested in the regularity and the qualitative behavior of the free boundary for small t. Here we consider only the model datum

$$g(x) = \begin{cases} p_+ x^\gamma, & x > 0 , \\ -p_-|x|^\gamma, & x < 0 , \end{cases} \tag{D}$$

where $p_\pm, \gamma > 0$ are given constants. For this case problem (P) has self-similar solutions which can be determined explicitly. The problem (P) with more general initial data will be included in a joint paper with J. Vazquez [HNV] where we use such self-similar solutions as comparison functions.

Proposition. For the model datum (D) problem $(\bar{P})$ has the unique self-similar solution

$$v(x,t) = t^{\gamma/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad x > s(t),\ t > 0$$

where $\psi(\cdot)$ is the unique solution of the ordinary differential equation (1) below, satisfying the initial conditions (2) and condition (3) at infinity. The free boundary is $s(t) = -\kappa\sqrt{t}$, $t > 0$. For $p := p_+/p_- > 0$ given, $\kappa > 0$ is the unique solution of the equation

$$p = \left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right) , \tag{E}$$

where $D_{-\gamma-1}(\cdot)$ is the parabolic cylinder function of index $(-\gamma - 1)$.

In the forthcoming paper with Vazquez [HNV] we will use the above Proposition to analyse (P) for more general constitutive functions and more general data; we assume there that ϕ is smooth on $[0,\infty)$ with $0 < c < \phi' < C$, $\phi(v) \equiv 0$ for $v < 0$ and that the more general data satisfy

$$G(x) = g(x) + o(|x|^\gamma) \qquad (|x| \to 0) ,$$

where g is the model datum. Then it will be shown that

$$s(t) = -\kappa\sqrt{\phi'(0^+)t} + o(\sqrt{t}) \qquad (t \searrow 0)$$

with κ defined as before. For $\phi(v) = \max(0,v)$ and $\gamma = 1$ we proved in [HN2,3] the stronger result

$$s(t) = -\kappa\sqrt{t} + O(t^{1/2+\alpha}) \qquad (t \searrow 0)$$

for any $\alpha < 1/2$. In this case (E) reduces to

$$p = \frac{\kappa^2}{2} + \frac{\kappa^3}{4}\exp\left(\frac{\kappa^2}{4}\right)\int_{-\kappa}^{\infty}\exp(-y^2/4)dy \; .$$

Note, that a first order expansion of the equation $g(s(t))s'(t) = v_x(s(t),t)$ in ($\bar{P}$) formally yields a different result, namely

$$(p_-)s(t)s'(t) = (p_+) + \cdots ,$$

which yields

$$s(t) = -\sqrt{2pt} + \cdots ,$$

whereas, e.g. for $p = 1$, $\kappa = .9034 \ldots \neq \sqrt{2}$. The reason for this apparent inconsistency is that all derivatives of v become singular at $(x,t) = (0,0)$; in particular v_x is not continuous at this point.

2. PROOF OF THE PROPOSITION. Substituting $v(x,t) = t^{\gamma/2}\psi(x/\sqrt{t})$ in (P) one sees that ψ must satisfy the linear differential equation

$$2\psi''(\xi) + \xi\psi'(\xi) - \gamma\psi(\xi) = 0 \quad \text{for} \quad \xi > -\kappa , \tag{1}$$

subject to the initial conditions

$$\psi(-\kappa) = 0, \; \psi'(-\kappa) = (p_-) \frac{\kappa^{\gamma+1}}{2} \tag{2}$$

and κ is related to $p_{\pm}$ via

$$\lim_{\xi\to+\infty} \xi^{-\gamma}\psi(\xi) = p_+ . \tag{3}$$

The free boundary is given by

$$s(t) = -\kappa\sqrt{t} .$$

Equation (1) can be solved explicitly. Put $x = \xi/\sqrt{2}$ and $w(\xi) = \psi(x)$. Then (1) becomes

$$w''(x) + xw'(x) - \gamma w(x) = 0 . \tag{4}$$

Setting $w(x) =: y(x)\exp(-x^2/4)$ we obtain

$$y''(x) - (\frac{1}{2} + \gamma + \frac{x^2}{4})y(x) = 0 . \tag{5}$$

This differential equation has the general solution [B, p. 116-117]

$$y(x) = b_1 D_{-\gamma-1}(x) + b_2 D_\gamma(ix) \quad (-\infty < x < \infty, \gamma > 0) ,$$

where $D_\nu(\cdot)$ is the parabolic cylinder function of index ν. Thus the general solution of (1) is

$$\psi(\xi) = [b_1 D_{-\gamma-1}(\frac{\xi}{\sqrt{2}}) + b_2 D_\gamma(\frac{i\xi}{\sqrt{2}})]\exp(\frac{-\xi^2}{8}) \tag{6}$$

for $-\infty < \xi < \infty$ and $\gamma > 0$. To impose the initial conditions (2) we need the formulae (above ref. p. 119)

$$\frac{d}{d\xi}\left[D_{-\gamma-1}\left(\frac{\xi}{\sqrt{2}}\right)\exp\left(\frac{-\xi^2}{8}\right)\right] = \frac{-1}{\sqrt{2}} D_{-\gamma}\left(\frac{\xi}{\sqrt{2}}\right)\exp\left(\frac{-\xi^2}{8}\right) ,$$

$$\frac{d}{d\xi}\left[D_{\gamma}\left(\frac{i\xi}{\sqrt{2}}\right)\exp\left(\frac{-\xi^2}{8}\right)\right] = \frac{i\gamma}{\sqrt{2}} D_{\gamma-1}\left(\frac{i\xi}{\sqrt{2}}\right)\exp\left(\frac{-\xi^2}{8}\right) .$$

Then the initial conditions (2) yield the pair of equations

$$\begin{aligned} &b_1 D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) + b_2 D_{\gamma}\left(-\frac{i\kappa}{\sqrt{2}}\right) = 0 \\ &-b_1 D_{\gamma}\left(-\frac{\kappa}{\sqrt{2}}\right) + i\gamma b_2 D_{\gamma-1}\left(-\frac{i\kappa}{\sqrt{2}}\right) = p_- \frac{\kappa^{\gamma+1}}{\sqrt{2}} \exp\left(\frac{\kappa^2}{8}\right) . \end{aligned} \tag{7}$$

Because (5) is of self-adjoint form the Wronskian of $D_{\gamma}(\cdot)$, $D_{-\gamma-1}(\cdot)$ is constant,

$$W(D_{-\gamma-1}(\cdot), D_{\gamma}(\cdot)) \equiv -i\exp\left[\left(\frac{\gamma+1}{2}\right)\pi i\right] .$$

Thus

$$\begin{aligned} b_1(\kappa) &= \frac{(p_-)\kappa^{\gamma+1} D_{\gamma}\left(-\frac{i\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right)}{i\sqrt{2}\exp\left[\left(\frac{\gamma+1}{2}\right)\pi i\right]} \\ b_2(\kappa) &= -\frac{(p_-)\kappa^{\gamma+1} D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right)}{i\sqrt{2}\exp\left[\left(\frac{\gamma+1}{2}\right)\pi i\right]} , \end{aligned} \tag{8}$$

and (6) with b_1, b_2 given by (8) is the solution of (1) satisfying the initial conditions (2). To compute the limit in (3) we use (see above ref. p. 122)

$$D_\nu(z) = z^\nu \exp\left(-\frac{z^2}{4}\right)[1 + O(|z|^{-2})] \quad \text{as} \quad |z| \to \infty , \tag{9}$$

which is valid for $-\frac{3\pi}{4} < \arg z < \frac{3\pi}{4}$. Thus for $\xi \in \mathbb{R}$, $\gamma > 0$,

$$D_{-\gamma-1}\left(\frac{\xi}{\sqrt{2}}\right) = \exp\left(-\frac{\xi^2}{8}\right)\left(\frac{\xi}{\sqrt{2}}\right)^{-\gamma-1}[1 + O(|\xi|^{-2})], \quad \xi \to +\infty ,$$

$$\left|D_\gamma\left(\frac{i\xi}{\sqrt{2}}\right)\right| = \exp\left(\frac{\xi^2}{8}\right)\left(\frac{\xi}{\sqrt{2}}\right)^{\gamma}\left|\exp\left(\frac{i\gamma\pi}{2}\right)\right|[1 + O(|\xi|^{-2})], \quad \xi \to +\infty . \tag{10}$$

Substitution of (10) and (8) into the general solution (6) yields

$$\psi(\xi) = b_2(\kappa)\exp\left(\frac{i\gamma\pi}{2}\right)\left(\frac{\xi}{\sqrt{2}}\right)^{\gamma}[1 + O(1)], \quad \xi \to +\infty . \tag{11}$$

From formula (8) we see that

$$b_2(k)\exp\left(\frac{i\gamma\pi}{2}\right) = \frac{p_-}{\sqrt{2}}\kappa^{\gamma+1}D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right) . \tag{12}$$

Imposing the asymptotic condition (3) and using (11), (12) we finally obtain

$$\lim_{\xi\to+\infty}\frac{\psi(\xi)}{\xi^\gamma} = p_+ = p_-\left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1}D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)\exp\left(\frac{\kappa^2}{8}\right) \tag{13}$$

which yields the equation (E).

To complete the proof of the Proposition we have to show that given any $p > 0$, (E) is uniquely solvable for κ. From [BO, p. 573]

$$D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) = \frac{\sqrt{\pi}}{2^{\frac{\gamma+1}{2}}\Gamma\left(1+\frac{\gamma}{2}\right)} \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!} \left(\frac{\kappa}{\sqrt{2}}\right)^{2n}$$

$$+ \frac{\sqrt{\pi}}{2^{\frac{\gamma}{2}}\Gamma\left(\frac{1+\gamma}{2}\right)} \sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n+1)!} \left(\frac{\kappa}{\sqrt{2}}\right)^{2n+1} , \tag{14}$$

is an analytic function of κ, $-\infty < \kappa < \infty$, $\gamma > -1$; $a_0 = a_1 = 1$, $a_{n+2} = \left(\gamma + \frac{1}{2}\right)a_n + \frac{n}{4}(n-1)a_{n-2}$, and $D_{-\gamma-1}(0) = \frac{\sqrt{\pi}}{2^{(\gamma+1)/2}\Gamma\left(1+\frac{\gamma}{2}\right)}$. Since the coefficients a_ν are positive, $D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right)$ is a positive, strictly increasing function of κ for $0 \leq \kappa < \infty$, and by (14) so is $p(\kappa)$. Moreover $p(0) = 0$. Also using (14) in (E)

$$p(\kappa) \approx \frac{\sqrt{\pi}}{2^{\frac{\gamma+1}{2}}\Gamma\left(1+\frac{\gamma}{2}\right)} \left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma+1} \qquad (\kappa \to 0^+) .$$

Moreover, [BO, p. 574]

$$D_{-\gamma-1}\left(-\frac{\kappa}{\sqrt{2}}\right) \approx \frac{\sqrt{2\pi}}{\Gamma(1+\gamma)} \left(\frac{\kappa}{\sqrt{2}}\right)^{\gamma} \exp\left(\frac{\kappa^2}{8}\right) \qquad (\kappa \to +\infty) ,$$

and therefore, from (E)

$$p(\kappa) \approx \frac{\sqrt{2\pi}}{\Gamma(1+\gamma)} \left(\frac{\kappa}{\sqrt{2}}\right)^{2\gamma+1} \exp\left(\frac{\kappa^2}{4}\right) \qquad (\kappa \to +\infty) .$$

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER 2728	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SELF SIMILAR SOLUTIONS FOR A DEGENERATE CAUCHY PROBLEM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Klaus Höllig and John A. Nohel		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE August 1984
		13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cauchy problem
degenerate
self-similar
free boundary
parabolic cylinder functions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We determine the self-similar solutions of the Cauchy problem

$$v_t = \phi(v)_{xx}, \quad x \in \mathbb{R},\ t > 0, \qquad v(x,0) = g(x) \tag{P}$$

for the constitutive function $\phi(v) = \max(0,v)$ and the model datum

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20. ABSTRACT - cont'd.

$$g(x) = \begin{cases} (p_+)x^\gamma, & x > 0 \\ -(p_-)|x|^\gamma, & x < 0 \end{cases} \tag{D}$$

where $\gamma, p_\pm > 0$. It is shown that the unique solution of (P), (D) is

$$v(x,t) = \begin{cases} -(p_-)|x|^\gamma, & x < -\kappa\sqrt{t} \\ t^{\gamma/2}\psi(\frac{x}{\sqrt{t}}), & x > -\kappa\sqrt{t} \end{cases}$$

where

$$\psi(\xi) = \lfloor b_1(\kappa)D_{-\gamma-1}(\frac{\xi}{\sqrt{2}}) + b_2(\kappa)D_\gamma(\frac{i\xi}{\sqrt{2}})\rfloor \exp(-\frac{\xi^2}{8}) ,$$

$$b_1(\kappa) = \frac{p_-\kappa^{\gamma+1}D_\gamma(-\frac{i\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8})}{i\sqrt{2}\exp(\frac{\gamma+1}{2}\pi i)} ,$$

$$b_2(\kappa) = \frac{p_-\kappa^{\gamma+1}D_{-\gamma-1}(-\frac{\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8})}{i\sqrt{2}\exp(\pi i\frac{\gamma+1}{2})}$$

and κ is implicitly determined by the equation

$$p := p_+/p_- = (\frac{\kappa}{\sqrt{2}})^{\gamma+1} D_{-\gamma-1}(-\frac{\kappa}{\sqrt{2}})\exp(\frac{\kappa^2}{8}) ;$$

$D_\nu(\cdot)$ denotes the parabolic cylinder function of index ν.

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